M.E. 530.647 Problem Set 0
Review of Simple Linear State Feedback Control

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1. Controller Design Review:

   Consider the following linear time-invariant plant with state vector $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \in \mathbb{R}^2$, input $u(t) \in \mathbb{R}^1$, and the following initial conditions at $t = 0$

   $$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t); \qquad x(0) = \begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix}. \tag{1}$$

   (a) Verify that this is the equation of motion for a unit mass constrained to 1-D motion on a straight line with position $x_1(t)$ and externally applied force $u(t)$.

   (b) Show that this system is controllable.

   (c) Design a state feedback controller $u = F(x)$ to stabilize the plant at the origin $x = [0, 0]^{\mathrm{T}}$ with closed-loop poles at $\{-2, -2\}$. Assume that the full state vector $x(t)$ is accessible. Show your work.

   (d) With the F you have computed, what is the natural frequency, ω_n, of this closed-loop system? What is the damping factor, ζ.

   (e) Prove that the state of closed-loop system converges to the origin, i.e. $\lim_{t\to\infty} x(t) = 0$.

2. Observer Design Review:

   Consider the linear time-invariant plant (1). Assume that you do not have access to the full state of the plant (1), you can only access the input $u(t)$ and the output $y(t)$ defined by

   $$y(t) = C\, x(t). \quad C = [1 \;\; 0]. \tag{2}$$

   (a) Show that this system is observable.

   (b) Design a Luenberger Observer to generate an on-line estimate, $\hat{x}(t)$ of xthe full state $x(t)$. Design the poles of your observer to be $\{-1, -1\}$. Show your work.

   (c) Prove that

   $$\lim_{t\to\infty} \Delta\hat{x}(t) = 0; \qquad \Delta\hat{x}(t) = \hat{x}(t) - x(t).$$

3. Controller/Observer Design Review: Construct a classical controller/observer feedback control system which asymptotically stabilizes the system (1), with output map (2), at the origin. Your design should join your state observer design (Question #2 of this problem set) together with the state feedback controller (Question #1 of this problem set). Assume that only the input $u(t)$ and the output $y(t)$ are accessible. Rather than employing the actual state (as you did in the previous problem set), your controller should instead employ the full state estimate $\hat{x}(t)$.

   (a) Derive your controller and observer and state it precisely.

   (b) Write the expression for the closed-loop dynamics of your system as the linear time-invariant differential equation of the form

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ \Delta\hat{x}(t) \end{bmatrix} = H \begin{bmatrix} x(t) \\ \Delta\hat{x}(t) \end{bmatrix} \tag{3}$$

   where the matrix H has block structure.

   (c) Prove that

$$\lim_{t\to\infty}\begin{bmatrix} x(t) \\ \Delta\hat{x}(t) \end{bmatrix} = 0. \tag{4}$$