M.E. 530.647 Problem Set 1

Revision 01

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https://dscl.lcsr.jhu.edu/courses/530-647-adaptive-systems-fall-2017

Refer to the text [5]. Additional useful references include [4, 1, 3, 6, 2]

1. Recall the definition of $\mathcal{L}_p$ functions from class (or refer to [5]). Consider the function $f : \mathbb{R}^1 \mapsto \mathbb{R}^1$

$$f(t) = \frac{1}{1+t}. \tag{1}$$

   (a) Is $f \in \mathcal{L}_1$?
   (b) Is $f \in \mathcal{L}_2$?
   (c) Is $f \in \mathcal{L}_\infty$?

2. Construct examples of each of the following. In each example identify the (i) state, (ii) input (if any), and (iii) parameters and also (iv) show that your example is a linear (or nonlinear) system.

   (a) Scalar first-order autonomous linear differential equation. Construct one example.
   (b) Scalar first-order non-autonomous linear differential equation. Construct one example.
   (c) Scalar first-order autonomous non-linear differential equation. Construct two examples.
   (d) Scalar first-order non-autonomous non-linear differential equation. Construct two examples.

3. Consider the first-order differential equation with initial value x_0 at $t = 0$

$$\dot{x}(t) = x(t)^2; \qquad x(0) = x_0; \quad x_0 > 0. \tag{2}$$

   (a) Solve for an exact solution $x(t)$ in terms of the independent variable t and the initial condition x_0.
   (b) Show that for $x_0 > 0$ the solution $x(t)$ becomes undefined in finite time.
   (c) At what t (as a function of x_0) does the solution fail to exist?
   (d) What conditions for existence and/or uniqueness of solutions to differential equations does this example violate?

4. Consider the first-order differential equation with initial value x_0 at $t = 0$

$$\dot{x}(t) = -|x(t)|^{\frac{1}{2}}; \qquad x(0) = x_0; \quad x_0 > 0. \tag{3}$$

   (a) Solve for an exact solution $x(t)$ in terms of the independent variable t and the initial condition $x_0 > 0$.

(b) Do solutions to this initial value problem exist for all initial conditions $x_0 >= 0$ for all time?

(c) Show that these solutions are not all unique for all time.

(d) What conditions for existence and/or uniqueness of solutions to differential equations does this example violate?

5. Consider the linear function $y : \mathbb{R}^1 \mapsto \mathbb{R}^1$ given by

$$y(t) = \beta\, u(t) \tag{4}$$

with output $y(u) \in \mathbb{R}^1$, input $u \in \mathbb{R}^1$, and unknown system parameter $\beta \in \mathbb{R}^1$.

(a) Suppose you have a single input-output pair $(y(t_1), u(t_1))$. Solve for β. Will this solution yield a valid value for β for all possible inputs $u \in \mathbb{R}^1$?

(b) Now suppose that the function arises from a real-world device with a random additive input noise of magnitude ϵ.

$$y(t) = \beta\, u(t) + \eta(t) \quad ||\eta(t)|| < \epsilon \tag{5}$$

You wish to estimate the unknown parameter β by asserting a measured input u_1 and measuring the resulting output. According to your instruments, the single measured input-output pair is

$$\{y(t_1), u(t_1))\}. \tag{6}$$

How does the *unknown* noise ϵ effect the accuracy of your estimate $\hat{\beta}$ in comparison to the true value of β in the following three cases:

i. $||u(t_1)|| >> \epsilon$.

ii. $||u(t_1)|| \approx \epsilon$.

iii. $||u(t_1)|| << \epsilon$.

(c) Suppose you have a set of input-output pairs $\{(y(t_i), u(t_i)\} \quad i \in \{1, \cdots, n\}$. Set this up as a least-square problem to estimate β. Under what condition on the $u(t_i)$ will this solution fail?

6. Consider the scalar first-order dynamical system

$$\dot{x} = \alpha\, x + \beta\, u \tag{7}$$

with state $x \in \mathbb{R}^1$, control input $u \in \mathbb{R}^1$, and unknown system parameters $\alpha \in \mathbb{R}^1$, $\beta \in \mathbb{R}^1$.

(a) Show that if you take several measurements of $\dot{x}(t), x(t)$, and $u(t)$

$$(\dot{x}(t_i), x(t_i), u(t_i)); \quad i \in 1, 2, \cdots, n \tag{8}$$

then the unknown parameters α and β can be estimated with the usual least-square methods. Under what conditions will this solution fail?

(b) Now suppose that you cannot measure $\dot{x}(t)$; you can only measure $x(t)$ and $u(t)$. Does the previous method for estimating α and β still work? Why or why not? Speculate (or demonstrate, if you can) how you might estimate α and β *without* differentiating $x(t)$.

7. Consider the first-order nonlinear autonomous dynamical system

$$\dot{x}(t) = -x(t)^3 \quad x(0) = x_0. \tag{9}$$

The initial condition x_0 may be positive or negative.

(a) Construct a linearized approximation to this system about the fixed point $x = 0$.

i. Is the resulting linear first order system stable?

ii. Is it asymptotically stable?

(b) Construct a strict global Lyapunov function for this system to show that it is globally asymptotically stable at the origin. Explicitly verify each part of the definition of a Lyapunov function.

References

[1] William E. Boyce and Richard C. DiPrima. *Elementary Differential Equations and Boundary Value Problems*. Wiley, New York, 1977.

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[5] K.S. Narendra and A. Annaswamy. *Stable Adaptive Systems*. Dover Publications, NY, 2005.

[6] Wilson J. Rugh. *Linear System Theory*. Prentice-Hall, New Jersey, 1995. (2nd Edition).