M.E. 530.647 Problem Set 2

Revision 01

Louis L. Whitcomb*
Department of Mechanical Engineering
G.W.C. Whiting School of Engineering
Johns Hopkins University
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1. Consider the second order nonlinear plant (a 1-D mechanical system with a mass m kg)

$$\dot{X}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} X(t) + \begin{bmatrix} 0 \\ m^{-1} \end{bmatrix} u(t); \tag{1}$$

with control input $u(t)$ Newtons and state vector

$$X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} \tag{2}$$

and the proportional derivative feedback control law

$$u(t) = -k_p x(t) - k_d \dot{x}(t) \tag{3}$$

where $k_p, k_d > 0$ are positive constant scalar feedback gains.

   (a) Write the equation for the closed loop system.

   (b) Use standard methods for linear systems to determine the stability properties of this system. Is it stable? Asymptotically stable?

   (c) Why is the scalar valued function

$$V(X,t) = \frac{1}{2} X^{\mathsf{T}} \begin{bmatrix} k_p & 0 \\ 0 & m \end{bmatrix} X \tag{4}$$

   not a strict global Lyapunov function for this system? Show your work checking each of the conditions required of a strict global Lyapunov function.

   (d) Using the Lyapunov function candidate from question 1c, invoke additional arguments to prove the global asymptotic stability of this system.

   (e) Construct a strict global Lyapunov function for this system.

2. Consider the initial value problem of the first-order differential equation

$$\dot{x}(t) = -u(t)^2 x(t); \quad x(0) = x_0. \tag{5}$$

(a) Show that the existence of constants $\epsilon > 0$ and $T > 0$ such that

$$\int_t^{t+T} u(\tau)^2 d\tau \geq \epsilon T \tag{6}$$

is a sufficient condition for the exponential convergence of $x(t)$ to zero.

(b) Show that this condition is also necessary.

3. Recall from the lecture that if $\lim_{t\to\infty} \int_0^t |f(\tau)|d\tau$ exists and is finite and if f is uniformly continuous, then $\lim_{t\to\infty} f(t) = 0$.

   (a) Show by example that *continuity* is an essential part of the above lemma by constructing a discontinuous function g for which $\lim_{t\to\infty} \int_0^t |g(\tau)|d\tau$ exists and is finite but for which $\lim_{t\to\infty} f(t) \neq 0$. Show these properties explicitly. (Hint: Try a square wave of unity magnitude and decreasing duty-cycle.)

   (b) Show by example that *uniform* continuity is an essential part of the above lemma by constructing a continuous function g for which $\lim_{t\to\infty} \int_0^t |g(\tau)|d\tau$ exists and is finite but for which $\lim_{t\to\infty} f(t) \neq 0$. Show these properties explicitly. (Hint: Modify the rectangular waveforms of the function for the previous question into triangles.)

4. Consider the linear algebraic function

$$y(t) = Au(t) \tag{7}$$

where $u(t) \in \mathbb{R}^n$ is a bounded, and continuous input signal, $A \in \mathbb{R}^{n\times n}$ is an unknown constant matrix, and $y(t) \in \mathbb{R}^n$ is a output signal. The signals $u(t)$ and $y(t)$ are available.

   (a) Given a set of times t_i and associated samples $u(t_i)$ and $y(t_i)$, construct an algebraic method to determine the unknown matrix A.

      i. Derive your solution and show that it satisfies the stated goal.
      ii. Under what conditions does your proposed solution succeed?
      iii. Under what conditions does your proposed solution fail?

   (b) Given continuous input $u(t)$ and output $y(t)$ signals, construct a non-adaptive method to use these continuous signals to determine the unknown matrix A.

      i. Derive your solution and show that it satisfies the stated goal.
      ii. Under what conditions does your proposed solution succeed?
      iii. Under what conditions does your proposed solution fail?

   (c) Given continuous input $u(t)$ and output $y(t)$ data, construct an adaptive method to determine the unknown matrix A.

      i. Derive your solution and show that it satisfies the stated goal.
      ii. Analyze the (i) stability properties of this method and (ii) the boundedness of all signals.
      iii. Under what conditions does your proposed method asymptotically exactly identify the unknown matrix A?
      iv. Under what conditions does your proposed method **not** asymptotically exactly identify the unknown matrix A?