Notes on Kronecker Products
Revision 03

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This note is a brief description of the matrix Kronecker product and matrix stack algebraic operators. For a more detailed treatment the reader is referred to [1, 2, 3].

1 The Stack Operator

The stack operator maps an \( n \times m \) matrix into an \( nm \times 1 \) vector. The stack of the \( n \times m \) matrix \( A \), denoted \( A^S \), is the vector formed by stacking the columns of \( A \) into an \( nm \times 1 \) vector.

For example if
\[
A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}_{2\times2}
\] (1)
then its stack form is
\[
A^S = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}_{4\times1}.
\] (2)

If \( C \) is an \( n \times m \) matrix comprising \( m \) column vectors \( \{c_1, c_2, \ldots, c_m\} \), where each \( c_i \) is an \( n \times 1 \) vector
\[
C = [c_1, c_2, \ldots, c_m]_{n \times m}
\] (3)
then
\[
C^S = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix}_{nm\times1}.
\] (4)

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1.1 Properties of the Stack Operator

1. If $v \in \mathbb{R}^{n \times 1}$, a vector, then $v^S = v$.

2. If $A \in \mathbb{R}^{m \times n}$, a matrix, and $v \in \mathbb{R}^{n \times 1}$, a vector, then the matrix product $(Av)^S = Av$.

3. $\text{trace}(AB) = ((A^T)^S)^T B^S$.

2 The Kronecker Product

The Kronecker product is a binary matrix operator that maps two arbitrarily dimensioned matrices into a larger matrix with special block structure. Given the $n \times m$ matrix $A_{n \times m}$ and the $p \times q$ matrix $B_{p \times q}$

$$
A = \begin{bmatrix}
a_{1,1} & \cdots & a_{1,m} \\
\vdots & \ddots & \vdots \\
a_{n,1} & \cdots & a_{n,m}
\end{bmatrix}_{n \times m}, \quad B = \begin{bmatrix}
b_{1,1} & \cdots & b_{1,q} \\
\vdots & \ddots & \vdots \\
b_{p,1} & \cdots & b_{p,q}
\end{bmatrix}_{p \times q}
$$

(5)

their Kronecker product, denoted $A \otimes B$, is the $np \times mq$ matrix with the block structure

$$
A \otimes B = \begin{bmatrix}
a_{1,1}B & \cdots & a_{1,m}B \\
\vdots & \ddots & \vdots \\
a_{n,1}B & \cdots & a_{n,m}B
\end{bmatrix}_{np \times mq}.
$$

(6)

For example, given

$$
A = \begin{bmatrix}1 & 2 \\ 0 & -1\end{bmatrix}_{2 \times 2}, \quad B_{2 \times 3} = \begin{bmatrix}1 & 2 & 3 \\ 4 & 5 & 6\end{bmatrix}_{2 \times 3}
$$

(7)

the Kronecker product $A \otimes B$ is

$$
A \otimes B = \begin{bmatrix}1 & 2 & 3 & 2 & 4 & 6 \\ 4 & 5 & 6 & 8 & 10 & 12 \\ 0 & 0 & 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & -4 & -5 & -6\end{bmatrix}_{4 \times 6}.
$$

(8)

3 Properties of the Kronecker Product and the Stack Operator

In the following it is assumed that $A$, $B$, $C$, and $D$ are real valued matrices. Some identities only hold for appropriately dimensioned matrices. For additional properties, see [1, 2, 3].

1. The Kronecker product is a bi-linear operator. Given $\alpha \in \mathbb{R}$,

$$
A \otimes (\alpha B) = \alpha (A \otimes B), \\
(\alpha A) \otimes B = \alpha (A \otimes B).
$$

(9)

2. Kronecker product distributes over addition:

$$
(A + B) \otimes C = (A \otimes C) + (B \otimes C), \\
A \otimes (B + C) = (A \otimes B) + (A \otimes C).
$$

(10)

3. The Kronecker product is associative:

$$
(A \otimes B) \otimes C = A \otimes (B \otimes C).
$$

(11)
4. The Kronecker product is not in general commutative, i.e. usually
   \[(A \otimes B) \neq (B \otimes A).\] (12)

5. Transpose distributes over the Kronecker product (does not invert order)
   \[(A \otimes B)^T = A^T \otimes B^T.\] (13)

6. Matrix multiplication, when dimensions are appropriate,
   \[(A \otimes B)(C \otimes D) = (AC \otimes BD).\] (14)

7. When \(A\) and \(B\) are square and full rank
   \[(A \otimes B)^{-1} = (A^{-1} \otimes B^{-1}).\] (15)

8. The determinant of a Kronecker product is (note right hand side exponents)
   \[det(A_{n \times n} \otimes B_{m \times m}) = det(A)^m \cdot det(B)^n.\] (16)

9. The trace of a Kronecker product is
   \[trace(A \otimes B) = trace(A) \cdot trace(B).\] (17)

10. Stack of a matrix multiplication, when dimensions are appropriate for the product \(ABC\) to be well defined, is
    \[(ABC)^S = (C^T \otimes A)B^S.\] (18)

    Two simple special cases of this identity are useful:
    \[Ax = (Ax)^S\] (19)
    and
    \[Ax = IAx = (x^T \otimes I)A^S\] (20)

11. If \(A\) and \(B\) are both
    (a) nonsingular
    (b) lower (upper) triangular
    (c) banded
    (d) symmetric
    (e) positive (negative) definite
    (f) stochastic
    (g) Toeplitz
    (h) permutation matrices
    (i) orthogonal

    then the product \(A \otimes B\) is, respectively, [2],
    (a) nonsingular
    (b) lower (upper) triangular
4 Some Applications

1. **The linear Lyapunov equation:** For any Hurwitz $A \in \mathbb{R}^{n \times n}$ and any positive-definite symmetric $Q \in \mathbb{R}^{n \times n}$ there exists a unique positive-definite symmetric $P \in \mathbb{R}^{n \times n}$ satisfying the linear Lyapunov equation

$$-Q = A^T P + PA. \quad (21)$$

The matrix $P$ can be computed as

$$P = \int_0^\infty e^{A^T \tau} Q e^{A \tau} d\tau \quad (22)$$

or, alternatively, note that

$$-Q = A^T P + PA = A^T P I + I P A \quad (23)$$

$$-Q^S = (I \otimes A^T) P^S + (A^T \otimes I) P^S$$

thus $P^S$ is given by

$$P^S = -(I \otimes A^T + A^T \otimes I)^{-1} Q^S. \quad (24)$$

2. **Differential matrix/vector calculus:** The derivative of vector-values and matrix-valued functions of vectors or matrices such as

$$y = A(b)c \quad (25)$$

where $y \in \mathbb{R}^n, b \in \mathbb{R}^m, c \in \mathbb{R}^p$, and $A : \mathbb{R}^m \rightarrow \mathbb{R}^{n \times p}$, can be written as

$$\dot{y} = D_x y = \frac{d}{dt}A(b)c = A(b)c + \frac{d}{dt}A(b)c = A(b)c + I \frac{d}{dt}A(b)c = A(b)c + (e^T \otimes I) \frac{d}{dt}A(b)c \quad (26)$$

where $D_x[y]$ is the usual matrix-valued jacobian operator given by

$$D_x[y] = \begin{bmatrix} \frac{\partial}{\partial x_1} y_1 & \cdots & \frac{\partial}{\partial x_1} y_1 \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial x_q} y_q & \cdots & \frac{\partial}{\partial x_q} y_q \end{bmatrix} \quad (27)$$

where $y \in \mathbb{R}^q, x \in \mathbb{R}^r$, and $D_x[y] \in \mathbb{R}^{n \times m}$. 
References

   https://store.doverpublications.com/0486824179.html


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