Notes on Kronecker Products

Revision 03

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This note is a brief description of the matrix Kronecker product and matrix stack algebraic operators. For a more detailed treatment the reader is referred to [1, 2, 3].

1 The Stack Operator

The stack operator maps an $n \times m$ matrix into an $nm \times 1$ vector. The stack of the $n \times m$ matrix A, denoted A^S, is the vector formed by stacking the columns of A into an $nm \times 1$ vector.

For example if

$$A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}_{2\times 2} \tag{1}$$

then its stack form is

$$A^S = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}_{4\times 1} . \tag{2}$$

If C is an $n \times m$ matrix comprising m column vectors $\{c_1, c_2, \cdots c_m\}$, where each c_i is an $n \times 1$ vector

$$C = [c_1, c_2, \cdots, c_m]_{n\times m} \tag{3}$$

then

$$C^S = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix}_{nm\times 1} . \tag{4}$$

1.1 Properties of the Stack Operator

1. If $v \in \mathbb{R}^{n\times 1}$, a vector, then $v^S = v$.
2. If $A \in \mathbb{R}^{m\times n}$, a matrix, and $v \in \mathbb{R}^{n\times 1}$, a vector, then the matrix product $(Av)^S = Av$.
3. $trace(AB) = ((A^T)^S)^T B^S$.

2 The Kronecker Product

The Kronecker product is a binary matrix operator that maps two arbitrarily dimensioned matrices into a larger matrix with special block structure. Given the $n \times m$ matrix $A_{n\times m}$ and the $p \times q$ matrix $B_{p\times q}$

$$A = \begin{bmatrix} a_{1,1} & \dots & a_{1,m} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \dots & a_{n,m} \end{bmatrix}_{n\times m} \qquad B = \begin{bmatrix} b_{1,1} & \dots & b_{1,q} \\ \vdots & \ddots & \vdots \\ b_{p,1} & \dots & b_{p,q} \end{bmatrix}_{p\times q} \tag{5}$$

their Kronecker product, denoted $A \otimes B$, is the $np \times mq$ matrix with the block structure

$$A \otimes B = \begin{bmatrix} a_{1,1}B & \dots & a_{1,m}B \\ \vdots & \ddots & \vdots \\ a_{n,1}B & \dots & a_{n,m}B \end{bmatrix}_{np\times mq} . \tag{6}$$

For example, given

$$A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}_{2\times 2} \qquad B_{2\times 3} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2\times 3} \tag{7}$$

the Kronecker product $A \otimes B$ is

$$A \otimes B = \begin{bmatrix} 1 & 2 & 3 & 2 & 4 & 6 \\ 4 & 5 & 6 & 8 & 10 & 12 \\ 0 & 0 & 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & -4 & -5 & -6 \end{bmatrix}_{4\times 6} . \tag{8}$$

3 Properties of the Kronecker Product and the Stack Operator

In the following it is assumed that A, B, C, and D are real valued matrices. Some identities only hold for appropriately dimensioned matrices. For additional properties, see [1, 2, 3].

1. The Kronecker product is a bi-linear operator. Given $\alpha \in \mathbb{R}$,

$$\begin{aligned} A \otimes (\alpha B) &= \alpha(A \otimes B) \\ (\alpha A) \otimes B &= \alpha(A \otimes B). \end{aligned} \tag{9}$$

2. Kronecker product distributes over addition:

$$\begin{aligned} (A + B) \otimes C &= (A \otimes C) + (B \otimes C) \\ A \otimes (B + C) &= (A \otimes B) + (A \otimes C). \end{aligned} \tag{10}$$

3. The Kronecker product is associative:

$$(A \otimes B) \otimes C = A \otimes (B \otimes C). \tag{11}$$

4. The Kronecker product is *not* in general commutative, i.e. usually

$$(A \otimes B) \neq (B \otimes A). \tag{12}$$

5. Transpose distributes over the Kronecker product (does *not* invert order)

$$(A \otimes B)^T = A^T \otimes B^T. \tag{13}$$

6. Matrix multiplication, when dimensions are appropriate,

$$(A \otimes B)(C \otimes D) = (AC \otimes BD). \tag{14}$$

7. When A and B are square and full rank

$$(A \otimes B)^{-1} = (A^{-1} \otimes B^{-1}). \tag{15}$$

8. The determinant of a Kronecker product is (note right hand side exponents)

$$det(A_{n\times n} \otimes B_{m\times m}) = det(A)^m \cdot det(B)^n. \tag{16}$$

9. The trace of a Kronecker product is

$$trace(A \otimes B) = trace(A) \cdot trace(B). \tag{17}$$

10. Stack of a matrix multiplication, when dimensions are appropriate for the product ABC to be well defined, is

$$(ABC)^S = (C^T \otimes A)B^S. \tag{18}$$

Two simple special cases of this identity are useful:

$$Ax = (Ax)^S \tag{19}$$

and

$$\begin{aligned} Ax &= IAx \\ &= (x^T \otimes I)A^S \end{aligned} \tag{20}$$

11. If A and B are both

(a) nonsingular
(b) lower (upper) triangular
(c) banded
(d) symmetric
(e) positive (negative) definite
(f) stochastic
(g) Toeplitz
(h) permutation matrices
(i) orthogonal

then the product $A \otimes B$ is, respectively, [2],

(a) nonsingular
(b) lower (upper) triangular

(c) block banded
(d) symmetric
(e) positive (negative) definite
(f) stochastic
(g) block Toeplitz
(h) a permutation matrix
(i) orthogonal.

4 Some Applications

1. **The linear Lyapunov equation:** For any Hurwitz $A \in \mathbb{R}^{n\times n}$ and any positive-definite symmetric $Q \in \mathbb{R}^{n\times n}$ there exists a unique positive-definite symmetric $P \in \mathbb{R}^{n\times n}$ satisfying the linear Lyapunov equation
$$-Q = A^T P + PA. \tag{21}$$
The matrix P can be computed as
$$P = \int_0^\infty e^{A^T\tau} Q e^{A\tau} d\tau \tag{22}$$
or, alternatively, note that
$$\begin{aligned} -Q &= A^T P + PA \\ &= A^T P I + I P A \\ -Q^S &= (I \otimes A^T)P^S + (A^T \otimes I)P^S \\ -Q^S &= (I \otimes A^T + A^T \otimes I)P^S \end{aligned} \tag{23}$$
thus P^S is given by
$$P^S = -(I \otimes A^T + A^T \otimes I)^{-1} Q^S. \tag{24}$$

2. **Differential matrix/vector calculus:** The derivative of vector-values and matrix-valued functions of vectors or matrices such as
$$y = A(b)c \tag{25}$$
where $y \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^p$, and $A : \mathbb{R}^m \mapsto \mathbb{R}^{n\times p}$, can be written as
$$\begin{aligned} \dot{y} &= \tfrac{d}{dt}[A(b)c] \\ &= A(b)\dot{c} + \tfrac{d}{dt}[A(b)]c \\ &= A(b)\dot{c} + I\tfrac{d}{dt}[A(b)]c \\ &= A(b)\dot{c} + [I\tfrac{d}{dt}[A(b)]c]^S \\ &= A(b)\dot{c} + (c^T \otimes I)\tfrac{d}{dt}[A(b)^S] \\ &= A(b)\dot{c} + (c^T \otimes I)D_b[A(b)^S]\dot{b} \end{aligned} \tag{26}$$
where $D_x[y]$ is the usual matrix-valued jacobian operator given by
$$D_x[y] = \begin{bmatrix} \frac{\partial}{\partial x_1} y_1 & \cdots & \frac{\partial}{\partial x_r} y_1 \\ \vdots & & \vdots \\ \frac{\partial}{\partial x_1} y_q & \cdots & \frac{\partial}{\partial x_r} y_q \end{bmatrix} \tag{27}$$
where $y \in \mathbb{R}^q$, $x \in \mathbb{R}^r$, and $D_x[y] \in \mathbb{R}^{n\times m}$.

References

[1] Alexander Graham. *Kronecker Products and Matrix Calculus With Applications*. Dover, NY, 1981. https://store.doverpublications.com/0486824179.html.

[2] Charles F Van Loan. The ubiquitous Kronecker product. *Journal of Computational and Applied Mathematics*, 123(1):85–100, 2000. https://www.sciencedirect.com/science/article/pii/S0377042700003939.

[3] Schäcke, Kathrin. On the Kronecker Product. Master's thesis, 2013. https://www.math.uwaterloo.ca/~hwolkowi/henry/reports/kronthesisschaecke04.pdf.